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A Monoidal Category of Bifinite Chu Spaces

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Abstract

Chu spaces are a general framework for studying the dualities of objects and properties, points and open sets, and terms and types, under rich mathematical contexts that are relevant to several sub-disciplines of computer science and mathematics. Traditionally, the study on Chu spaces had a “non-constructive” flavor. The recent work of Droste and Zhang [4] on bifinite Chu spaces provides a basis for a constructive analysis of Chu spaces and opens the door to a more systematic investigation of such an analysis in a variety of settings. As a step in this direction, we show in this paper that a category of bifinite Chu spaces is monoidal, but not monoidal closed.

Keywords: Chu space, colimit, E-bifinite, monoidal category, non-monoidal-closedness

1 Introduction

In [4], Droste and Zhang introduced a special category of Chu spaces called bifinite Chu spaces. Bifinite Chu spaces can be viewed, in an intuitive category-theoretical sense, as “countable” objects which are approximable by the finite objects of the

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category. Such a notion of structural approximation is captured categorically as the colimit, with the underlying morphisms being monics.

The main objectives of [4] were to characterize monics in Chu spaces, formulate and instantiate the notion of colimits, characterize finite objects in the category of Chu spaces with monics. Basic properties of bifinite Chu spaces were also investigated, including the existence of a universal, homogeneous object.

This paper studies the monoidal closedness properties of bifinite Chu spaces, and introduces a new notion called E-bifinite Chu spaces. These are defined in the category of extensional Chu spaces with monics, as colimits of ω -sequences of finite spaces. All bifinite Chu spaces are E-bifinite. We show that the category of E-bifinite Chu spaces is monoidal, but not monoidal closed. The key construction for a monoidal category is the tensor product. We use the standard definition of the tensor product in Chu spaces [3] and show that the category of bifinite Chu spaces with monics is not closed under tensor products, but the same tensor product induces a continuous bi-functor with respect to E-bifinite Chu spaces. Together with an appropriately defined unit, we obtain a monoidal category. A proof is given to show that this monoidal category is, however, not monoidal closed. The intuitive reason for this negative result is that the colimit construction is not symmetric with respect to objects and attributes in Chu spaces, and so linear negation (“perp” or transposition) is not a functor anymore: it can reduce an E-bifinite Chu space to a non E-bifinite one.

2 Chu spaces

We recall some fundamental definitions and notions directly concerned with Chu spaces, following [4].

Definition 2.1 A Chu space \mathcal{C}_Σ over a set Σ is a triple (A, r, X) , where A is a set whose elements are generally considered as objects and X a set whose elements can be regarded as attributes on A . The satisfaction relation r is a function $A \times X \rightarrow \Sigma$ which describes attributes of every object in A . A morphism from a Chu space $\mathcal{A}_\Sigma = (A, r, X)$ to a Chu space $\mathcal{B}_\Sigma = (B, s, Y)$ is a pair of function $\varphi = (f, g)$ with $f : A \rightarrow B$ and $g : Y \rightarrow X$ such that for any $a \in A$ and $y \in Y$, $r(a, g(y)) = s(f(a), y)$.

In the following, when considering a class of Chu spaces, we always assume that they are based on the same set Σ and so leave it unspecified, simply denoting \mathcal{C}_Σ by \mathcal{C} . Following the convention of category theory, we denote the set of morphisms from the Chu space \mathcal{A} to \mathcal{B} as $\text{Hom}(\mathcal{A}, \mathcal{B})$. In addition, we refer to the forward connecting of a morphism $\varphi = (f, g)$ by $\varphi^+ = f$ and the backward connecting by $\varphi^- = g$. For sake of convenience, we denote the object set A of \mathcal{C} by $\text{obj}(\mathcal{C})$ and the attribute set X by $\text{attr}(\mathcal{C})$.

For any two morphisms $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ and $\psi : \mathcal{B} \rightarrow \mathcal{C}$, define the composition of φ with ψ as $\varphi \circ \psi = (\psi^+ \cdot \varphi^+, \varphi^- \cdot \psi^-)$. It can be readily checked that $\varphi \circ \psi$ is a morphism from \mathcal{A} to \mathcal{C} . Now let \mathbf{C} denote the category of Chu spaces with

morphisms and composition defined as above.

Given a Chu space $\mathcal{C} = (A, r, X)$ over the set Σ , its dual is $\mathcal{C}^\perp = (X, r^\perp, A)$ where $r^\perp : X \times A \rightarrow \Sigma$ is defined by $r^\perp(x, a) = r(a, x)$ for any $(x, a) \in X \times A$. For any morphism $\varphi : \mathcal{A} \rightarrow \mathcal{B}$, $\varphi^\perp = (\varphi^-, \varphi^+) : \mathcal{B}^\perp \rightarrow \mathcal{A}^\perp$. Thus, \perp is a contravariant functor on \mathbf{C} .

Since the satisfaction relation r of a Chu space $\mathcal{A} = (A, r, X)$ is a function from $A \times X$ to Σ , we can regard it as a table with rows on A , columns on X and values on Σ . Then we can define two kinds of equivalence relations in a natural way. One is on the rows, where the a -th row corresponds to the function $r(a, -) : A \rightarrow \Sigma$. Two rows of a and b are *equivalent* if $r(a, -) = r(b, -)$, that is, they are repeated lines in the table of r . Similarly, equivalence can be defined on columns, by equality $r(-, x) = r(-, y)$ entailing $x = y$. A Chu space (A, r, X) is called *extensional* if $r(-, x) = r(-, y)$ implies $x = y$, i.e., r does not contain repeated columns. Similarly, a Chu space (A, r, X) is *separable* if it does not contain repeated rows. In topological analogy, if we think of objects in A as points and attributes in X as open sets, then separable Chu spaces are those for which distinct points can be differentiated by the open sets containing them (such spaces are called T_0 in Topology). A Chu space is *biextensional* if it is both separable and extensional. The full subcategory of \mathbf{C} with extensional (biextensional) Chu spaces as objects is denoted by \mathbf{E} and \mathbf{B} respectively.

In categorical terms, a morphism $\psi : C_1 \rightarrow C_2$ is monic if for any other two morphisms $\psi_i : C_3 \rightarrow C_1$ ($i = 1, 2$) such that $\psi \circ \psi_1 = \psi \circ \psi_2$, we have $\psi_1 = \psi_2$. In particular, monics on the category of Chu spaces are characterized in [4] as follows.

Proposition 2.2 *In the category of Chu spaces, we have*

- (1) *A morphism $\psi : \mathcal{A} \rightarrow \mathcal{B}$ in \mathbf{C} is monic if and only if ψ^+ is injective and ψ^- is surjective.*
- (2) *A morphism $\psi : \mathcal{A} \rightarrow \mathcal{B}$ in \mathbf{E} is monic if and only if ψ^+ is injective.*
- (3) *A morphism $\psi : \mathcal{A} \rightarrow \mathcal{B}$ in \mathbf{B} is monic if and only if ψ^+ is injective.*

We denote the subcategories of \mathbf{C} , \mathbf{E} and \mathbf{B} with monics as homomorphisms by $i\mathbf{C}$, $i\mathbf{E}$ and $i\mathbf{B}$, respectively. Note that although \mathbf{E} and \mathbf{B} are full subcategories of \mathbf{C} , it is not the case for $i\mathbf{E}$ or $i\mathbf{B}$ relative to $i\mathbf{C}$.

Corollary 2.3 *A morphism $\psi : \mathcal{A} \rightarrow \mathcal{B}$ in \mathbf{C} is an isomorphism if and only if both ψ^+ and ψ^- are bijective.*

Proof. It is obvious that ψ will be an isomorphism if both ψ^+ and ψ^- are bijective. Conversely, assume that ψ is an isomorphism. Then from Proposition 2.2, we know that ψ^+ is injective and ψ^- is surjective. Meanwhile, since $\psi^\perp : \mathcal{B} \rightarrow \mathcal{A}$ is also an isomorphism, $(\psi^\perp)^+$ is injective and $(\psi^\perp)^-$ is surjective. While $(\psi^\perp)^+ = \psi^-$ and $(\psi^\perp)^- = \psi^+$, both ψ^+ and ψ^- are then bijective. \square

We recall the well-known tensor product in the category of Chu space.

Definition 2.4 The tensor functor $\otimes : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$ in the category \mathbf{C} of Chu spaces is defined as ([3]):

- (1) For any two Chu spaces $\mathcal{A} = (A, r, X)$ and $\mathcal{B} = (B, s, Y)$, define $\mathcal{A} \otimes \mathcal{B} = (A \times B, t, \text{Hom}(\mathcal{A}, \mathcal{B}^\perp))$, where for any $(a, b) \in \text{obj}(\mathcal{A} \otimes \mathcal{B})$ and $\varphi \in \text{attr}(\mathcal{A} \otimes \mathcal{B})$, $t((a, b), \varphi) = r(a, \varphi^-(b))$ (also equals $s(b, \varphi^+(a))$);
- (2) For any two morphisms $\varphi : \mathcal{A} \rightarrow \mathcal{A}'$ and $\psi : \mathcal{B} \rightarrow \mathcal{B}'$, $\varphi \otimes \psi : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{A}' \otimes \mathcal{B}'$ is defined by:

$$(\varphi \otimes \psi)^+(a, b) = (\varphi^+(a), \psi^+(b)) \text{ for any } (a, b) \in \text{obj}(\mathcal{A} \otimes \mathcal{B});$$

$$(\varphi \otimes \psi)^-(f', g') = (\psi^- \cdot f' \cdot \varphi^+, \varphi^- \cdot g' \cdot \psi^+) \text{ for any } (f', g') \in \text{attr}(\mathcal{A}' \otimes \mathcal{B}').$$

According to the definition of homomorphisms in the category of Chu spaces, we know that the object set $\text{Hom}(\mathcal{A}, \mathcal{B}^\perp)$ in Definition 2.4 is just the collection of connecting morphisms $(f : A \rightarrow Y, g : B \rightarrow X)$ from \mathcal{A} to \mathcal{B}^\perp such that $r(a, g(b)) = s(b, f(a))$ for any $a \in A$ and $b \in B$.

Proposition 2.5 *The tensor product of two extensional Chu spaces is extensional.*

Proof. Let $\mathcal{A} = (A, r, X), \mathcal{B} = (B, s, Y)$ be two extensional Chu spaces, and $(f, g), (f', g') \in \text{attr}(\mathcal{A} \otimes \mathcal{B})$ be distinct. Then we have $f \neq f'$. Otherwise, $g \neq g'$ and it follows that there exists an element $b \in B$ such that $g(b) \neq g'(b)$. Since \mathcal{A} is extensional, there exists an element a with $r(a, g(b)) \neq r(a, g'(b))$. While $s(b, f(a)) = r(a, g(b))$, $s(b, f'(a)) = r(a, g'(b))$ and $s(b, f(a)) = s(b, f'(a))$, a contradiction arises. So f cannot be equal to f' .

Since $f \neq f'$, there exists $a \in A$ with $f(a) \neq f'(a)$. Since \mathcal{B} is extensional, there exists $b \in B$ with $s(b, f(a)) \neq s(b, f'(a))$. Thus $t((a, b), (f, g)) = s(b, f(a)) \neq s(b, f'(a)) = t((a, b), (f', g'))$. \square

The tensor product of two monics in \mathbf{C} may not be monic any more. Furthermore, the tensor product behaves quite differently in distinct subcategories of \mathbf{C} . For example, in the category $i\mathbf{E}$, the tensor product of two extensional Chu spaces remains to be extensional, and from Proposition 2.2, we know that the tensor product of two monics is still a monic. However, in the category \mathbf{B} tensor product of two biextensional Chu spaces may not remain biextensional, and this can be shown by the fact that the attribute set of the tensor product may be empty, as given in the next example.

Example 2.6 Let $\Sigma = \{0, 1\}$, A, B, A', X, X', Y and Y' are all $\{0\}$, $B' = \{0, 1\}$; let r, r' and s be maps from $A \times X, B \times Y$ and $A' \times X'$ to $\{0\}$, respectively. s' maps $B' \times Y'$ to Σ by $s'(0, 0) = 0$ and $s'(1, 0) = 1$. Let $\mathcal{A} = (A, r, X), \mathcal{B} = (B, s, Y), \mathcal{A}' = (A', r', X')$ and $\mathcal{B}' = (B', s', Y')$. Further, let $\varphi = (id, id) : \mathcal{A} \rightarrow \mathcal{A}'$ and $\psi = (e, id) : \mathcal{B} \rightarrow \mathcal{B}'$, where id denotes the identity mapping and e is the embedding mapping from $\{0\}$ to $\{0, 1\}$. It is clear that both φ and ψ are monics in \mathbf{C} , but the tensor product $\varphi \otimes \psi$ is not since $\text{attr}(\mathcal{A} \otimes \mathcal{B})$ has one element while $\text{attr}(\mathcal{A}' \otimes \mathcal{B}')$ is empty.

Given the set Σ , we define a special Chu space $\mathcal{I} = (\{\star\}, \tau, \Sigma)$, where $\tau : \{\star\} \times \Sigma \rightarrow \Sigma$ satisfies $\tau(\star, x) = x$ for any $x \in \Sigma$. Then \mathcal{I} is obviously biextensional. Further, \mathcal{I} has the unital property.

Proposition 2.7 \mathcal{I} is the unit object in \mathbf{C} , that is, for any Chu space \mathcal{A} , $\mathcal{A} \otimes \mathcal{I} \cong \mathcal{A}$ and $\mathcal{I} \otimes \mathcal{A} \cong \mathcal{A}$ with respect to natural isomorphisms.

Proof. First we prove that given a Chu space \mathcal{A} , $\mathcal{A} \otimes \mathcal{I} \cong \mathcal{A}$ with respect to natural isomorphisms. In fact, let $\varphi_{\mathcal{A}}^+ : \text{obj}(\mathcal{A} \otimes \mathcal{I}) \rightarrow \text{obj}(\mathcal{A})$ be: for any $(a, \star) \in \text{obj}(\mathcal{A} \otimes \mathcal{I})$, $\varphi_{\mathcal{A}}^+(a, \star) = a$; let $\varphi_{\mathcal{A}}^- : \text{attr}(\mathcal{A}) \rightarrow \text{attr}(\mathcal{A} \otimes \mathcal{I})$ be: for any $x \in \text{attr}(\mathcal{A})$, $\varphi_{\mathcal{A}}^-(x) = (r(-, x), c_x)$, where r is the satisfaction relation on \mathcal{A} and c_x is the constant function onto x . Then it can be checked that $(\varphi_{\mathcal{A}}^+, \varphi_{\mathcal{A}}^-)$ forms an isomorphism from $\mathcal{A} \otimes \mathcal{I}$ to \mathcal{A} . Furthermore, it can be checked that the diagram

$$\begin{array}{ccc} \mathcal{A} \otimes \mathcal{I} & \xrightarrow{\varphi_{\mathcal{A}}} & \mathcal{A} \\ \psi \otimes 1_{\mathcal{I}} \downarrow & & \downarrow \psi \\ \mathcal{A}' \otimes \mathcal{I} & \xrightarrow{\varphi_{\mathcal{A}'}} & \mathcal{A}' \end{array}$$

commutes for any morphisms ψ from one Chu space \mathcal{A} to another \mathcal{A}' . Thus $\varphi_{\mathcal{A}}$ is natural.

Likewise, we can prove that $\mathcal{I} \otimes \mathcal{A} \cong \mathcal{A}$ with respect to natural isomorphisms. Therefore, \mathcal{I} is the unit object in \mathbf{C} . \square

With the dual operator \perp and the tensor product \otimes , the linear operator \multimap in the category \mathbf{C} of Chu spaces is defined as ([3]):

Definition 2.8 The linear operator $\multimap : \mathbf{C}^{op} \times \mathbf{C} \rightarrow \mathbf{C}$ is:

- (1) for any two Chu spaces \mathcal{A} and \mathcal{B} , $\mathcal{A} \multimap \mathcal{B} = (\mathcal{A} \otimes \mathcal{B}^\perp)^\perp = (\text{Hom}(\mathcal{A}, \mathcal{B}), t, \text{obj}(\mathcal{A}) \times \text{attr}(\mathcal{B}))$;
- (2) for any two morphisms $\varphi : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ and $\psi : \mathcal{B}_1 \rightarrow \mathcal{B}_2$, $\varphi \multimap \psi = (\varphi \otimes \psi^\perp)^\perp : (\mathcal{A}_2 \otimes \mathcal{B}_1^\perp)^\perp \rightarrow (\mathcal{A}_1 \otimes \mathcal{B}_2^\perp)^\perp$ satisfies:

$$\text{for any } \alpha \in \text{obj}((\mathcal{A}_2 \otimes \mathcal{B}_1^\perp)^\perp), (\varphi \multimap \psi)^+(\alpha) = (\psi^+ \cdot \alpha^+ \cdot \varphi^+, \varphi^- \cdot \alpha^- \cdot \psi^-);$$

$$\text{for any } (a_1, y_2) \in \text{attr}((\mathcal{A}_1 \otimes \mathcal{B}_2^\perp)^\perp), (\varphi \multimap \psi)^-(a_1, y_2) = (\varphi^+(a_1), \psi^-(y_2)).$$

We shall see that such a popular linear operator, when combined with the tensor product defined above, will be troublesome for obtaining monoidal closedness.

3 E-bifinite Chu spaces

The notion of bifinite Chu spaces was introduced in [4], based on fundamental notions of ω -sequences, colimits, and finite objects of Chu spaces. In categorical terms, bifinite Chu spaces can be viewed as “countable” objects which are approximated by finite objects of Chu Spaces. In [4], many interesting results were developed. In this section, we introduce a new kind of Chu spaces, named E-bifinite Chu spaces, which are similar to, yet distinct from bifinite Chu spaces. In order to define E-bifinite Chu spaces, we need some preparations first.

Definition 3.1 An ω -sequence in $i\mathbf{C}$ is a family $(\mathcal{C}_i, \varphi_i)_{i \geq 1}$, where $\varphi_i : \mathcal{C}_i \rightarrow \mathcal{C}_{i+1}$ is monic for each $i \geq 1$.

$$\mathcal{C}_1 \xrightarrow{\varphi_1} \mathcal{C}_2 \xrightarrow{\varphi_2} \mathcal{C}_3 \xrightarrow{\varphi_3} \cdots \xrightarrow{\varphi_i} \mathcal{C}_{i+1} \cdots$$

Definition 3.2 A cone from an ω -sequence $(\mathcal{C}_i, \varphi_i)_{i \geq 1}$ to a Chu space $\mathcal{C} = (A, r, X)$ is a family of morphisms $\psi_i : \mathcal{C}_i \rightarrow \mathcal{C}$ such that $\psi_{i+1} \circ \varphi_i = \psi_i$, for all $i \geq 1$, i.e., the diagram

$$\begin{array}{ccccccc} \mathcal{C}_1 & \xrightarrow{\varphi_1} & \mathcal{C}_2 & \xrightarrow{\varphi_2} & \mathcal{C}_3 & \xrightarrow{\varphi_3} & \cdots \xrightarrow{\varphi_i} \mathcal{C}_{i+1} \cdots \\ & \searrow \psi_1 & \searrow \psi_2 & \searrow \psi_3 & \searrow \psi_{i+1} & & \\ & & & \downarrow & & & \\ & & & \mathcal{C} & & & \end{array}$$

commutes.

A cone $(\mathcal{C}, (\psi_i : \mathcal{C}_i \rightarrow \mathcal{C})_{i \geq 1})$ is universal if for any other cone $(\mathcal{C}', (\psi'_i : \mathcal{C}_i \rightarrow \mathcal{C}')_{i \geq 1})$ such that $\psi'_{i+1} \circ \varphi_i = \psi'_i$ for all $i \geq 1$, there exists a unique morphism $\psi : \mathcal{C} \rightarrow \mathcal{C}'$ such that $\psi \circ \psi_i = \psi'_i$ for all $i \geq 1$. Such a universal cone, if exists, is called the colimit of the family $(\mathcal{C}_i, \varphi_i)_{i \geq 1}$ and denoted by $\text{colim}_{i \geq 1} \mathcal{C}_i$. ψ is called the mediating map.

As had been pointed out in [4], ω -sequence of Chu spaces may have no colimit in the category $i\mathbf{C}$. However, when colimits exist, we can provide an explicit construction, which is the standard construction in the category of sets, assimilated in the context of Chu spaces.

Construction 3.3 Let $(\mathcal{C}_i, \varphi_i)_{i \geq 1}$ be an ω -sequence of Chu spaces where $\mathcal{C}_i = (A_i, r_i, X_i)$ and $\varphi_i^+ : A_i \rightarrow A_{i+1}$ is the inclusion mapping, for each $i \geq 1$. Consider $\mathcal{C} = (A, r, X)$ where

$$\begin{aligned} A &= \bigcup_{i \geq 1} A_i, \\ X &= \{(x_j)_{j \geq 1} \mid \forall j \geq 1, x_j \in X_j \text{ \& } \varphi_j^-(x_{j+1}) = x_j\}, \\ r(a, (x_j)_{j \geq 1}) &= r_i(a, x_i) \text{ if } a \in A_i \text{ } (i \geq 1). \end{aligned}$$

Subsequently, we denote a sequence $(x_j)_{j \geq 1} \in X$ often by \tilde{x} .

For each $i \geq 1$, define $\pi_i : \mathcal{C}_i \rightarrow \mathcal{C}$ by $\pi_i^+(a) = a$ and $\pi_i^-(\tilde{x}) = x_i$ for all $a \in A_i$ and $\tilde{x} \in X$.

The satisfaction relation r is well-defined since if $i \geq 1$ and $a \in A_i$, then $x_i = \varphi_i^-(x_{i+1})$ so $r_{i+1}(a, x_{i+1}) = r_i(a, x_i)$; inductively we obtain $r_j(a, x_j) = r_i(a, x_i)$ for each $j > i$. Note that it is possible for the set $X1$ to be empty.

Clearly, π_i is a morphism in the category of Chu spaces. For each $\tilde{x} \in X$,

$$(\varphi_i^- \circ \pi_{i+1}^-)(\tilde{x}) = \varphi_i^-(x_{i+1}) = x_i = \pi_i^-(\tilde{x}).$$

And for any $a \in A$,

$$(\pi_{i+1}^+ \circ \varphi_i^+)(a) = a = \pi_i^+(a).$$

Therefore, $\pi_{i+1} \circ \varphi_i = \pi_i$, and $(\varphi_i : \mathcal{C}_i \rightarrow \mathcal{C})_{i \geq 1}$ is indeed a cone.

Such a construction gives the standard formulation of colimits in various categories.

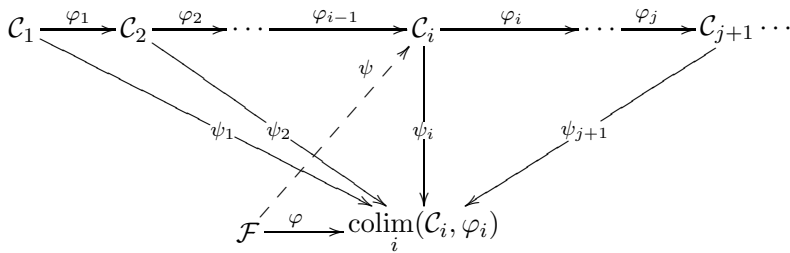
Proposition 3.4 *If an ω -sequence of Chu spaces in $i\mathbf{C}$ has colimits, then they are isomorphic to the cone of Construction 3.3.*

Theorem 3.5 *Colimits exist in $i\mathbf{E}$, as given in Construction 3.3.*

By finite Chu structures, we mean that they are Chu spaces with finite object sets and finite attribute sets.

Theorem 3.6 *Colimits exist in $i\mathbf{B}$ for ω -sequences of finite Chu structures and are isomorphic to the cone given in Construction 3.3.*

Definition 3.7 An object \mathcal{F} of $i\mathbf{C}$ ($i\mathbf{E}$, $i\mathbf{B}$) is finite if for every ω -sequence $(\mathcal{C}_i, \varphi_i)_{i \geq 1}$ of Chu spaces having a colimit in $i\mathbf{C}$ ($i\mathbf{E}$, $i\mathbf{B}$, respectively), for every morphism $\varphi : \mathcal{F} \rightarrow \text{colim}_{i \geq 1} \mathcal{C}_i$ in $i\mathbf{C}$ ($i\mathbf{E}$, $i\mathbf{B}$, respectively) there exist an $i \geq 1$ and a morphism $\psi : \mathcal{F} \rightarrow \mathcal{C}_i$ such that $\varphi = \psi_i \circ \psi$. Namely, the diagram following commutes.



Whether a Chu space is a finite object in $i\mathbf{C}$ or not can be checked with the following theorem given in [4].

Theorem 3.8 *An object $\mathcal{F} = (B, s, Y)$ is finite in $i\mathbf{C}$ if and only if B is finite and \mathcal{F} is extensional.*

With this theorem, we know that finite objects are tensor multiplicative.

Corollary 3.9 *The tensor product of two finite objects in $i\mathbf{C}$ is a finite object in $i\mathbf{C}$.*

Proof. Let \mathcal{A} and \mathcal{B} be two finite objects in $i\mathbf{C}$. By the preceding theorem, $\text{obj}(\mathcal{A})$ and $\text{obj}(\mathcal{B})$ are finite, and so $\text{obj}(\mathcal{A} \otimes \mathcal{B}) = \text{obj}(\mathcal{A}) \times \text{obj}(\mathcal{B})$ is finite. In addition, $\mathcal{A} \otimes \mathcal{B}$ is extensional since so are both \mathcal{A} and \mathcal{B} . Thus again from the preceding theorem, we have $\mathcal{A} \otimes \mathcal{B}$ is finite in $i\mathbf{C}$. \square

Definition 3.10 A Chu space (A, r, X) over Σ is called complete, if for any map $f : A \rightarrow \Sigma$ there is $x \in X$ such that $f = r(-, x)$.

Note that such completeness of Chu spaces is not tensor multiplicative, namely, the tensor product of two complete Chu spaces may not be complete. This can be

readily observed when the attribute set of the tensor product of two Chu spaces is empty.

By Theorem 3.8, Chu spaces having finite object sets and empty attribute sets are finite objects in $i\mathbf{C}$, $i\mathbf{E}$ and $i\mathbf{B}$. Further,

Theorem 3.11 *In both categories $i\mathbf{E}$ and $i\mathbf{B}$, a Chu space \mathcal{F} with non-empty attribute set is a finite object if and only if it is complete and its object set $\text{obj}(\mathcal{F})$ is finite.*

A Chu space \mathcal{F} is called *strongly finite*, if $\mathcal{F} = (A, r, X)$ is both a finite object in $i\mathbf{C}$ and a finite Chu structure. Here, the attribute set of a strongly finite Chu space is not required to be non-empty as in [4]. By Theorem 3.8, a Chu space is strongly finite if and only if it is extensional and has both finite object set and finite attribute set.

Definition 3.12 A Chu space is called E-bifinite if it is isomorphic to the colimit (with respect to $i\mathbf{E}$) of an ω -sequence of strongly finite objects in $i\mathbf{E}$. The corresponding full subcategory of E-bifinite Chu spaces of \mathbf{E} and $i\mathbf{E}$ are denoted as \mathbf{E}_{bif} and $i\mathbf{E}_{bif}$, respectively.

By definition, all strongly finite Chu spaces are E-bifinite. Moreover, all bifinite Chu spaces in the sense of [4] are also E-bifinite.

Here, we would like to point out the difference between the notions of E-bifinite Chu spaces just defined and bifinite Chu spaces as given in Definition 5.1 of [4]. In both cases, the objects of the ω -sequences considered are strongly finite spaces, but in Definition 3.12 the connecting morphisms in the ω -sequence are monics in \mathbf{E} , whereas in [4] they are monics in \mathbf{C} : refer again to Proposition 2.2 for the difference. Hence, formally, \mathbf{E}_{bif} contains all bifinite Chu spaces (in the sense of [4]). More importantly, we have seen in Example 2.6 that the tensor product of two monics in \mathbf{C} may not be monic anymore. However, the tensor product of two monics in \mathbf{E} is monic again, and this is the very reason for our new definition of E-bifinite spaces.

In this context we note that in [4] and in this paper we could have replaced ω -sequences of Chu spaces by co-directed systems of Chu spaces.

A family of Chu spaces $\{\mathcal{C}_i\}_{i \in I}$ with monics $\varphi_{i,j} : \mathcal{C}_i \rightarrow \mathcal{C}_j$ whenever $i \leq j$ is a co-directed system if:

- the index set I is directed, that is, for any $i, j \in I$, there exists $k \in I$ such that $i \leq k$ and $j \leq k$;
- $\varphi_{i,i}$ is the identity on \mathcal{C}_i for any $i \in I$;
- $\varphi_{i,k} = \varphi_{j,k} \cdot \varphi_{i,j}$ for any $i \leq j \leq k$.

With a co-directed system of Chu spaces, we can readily reformulate most of notions introduced in [4], such as cone, colimit, finiteness, completeness, in the same manner. Further, a similar construction as Construction 3.3 can be given as follows.

Construction 3.13 *Let $\{\varphi_{i,j} : \mathcal{C}_i \rightarrow \mathcal{C}_j\}_{i \leq j}$ be a co-directed system of Chu spaces in $i\mathbf{C}$, where, $\mathcal{C}_i = (A_i, r_i, X_i)$. Let the set $\tilde{A} = \{\{a_i\}_{i \geq i_0} : i_0 \in I, a_{i_0} \in$*

A_i , and for any $j \geq i_0, a_j = \varphi_{i_0,j}^+(a_{i_0})\}$. In addition, define a binary relation \sim on \tilde{A} as: $\{a_i\}_{i \geq i_0} \sim \{a'_i\}_{i \geq j_0}$ iff there exists an $k \geq i_0, j_0$ such that $a_k = a'_k$. Then \sim is an equivalence relationship on \tilde{A} . Denote every equivalent class of $\{a_i\}_{i \geq i_0}$ by $[a_{i_0}]$. Then construct the Chu space $\mathcal{C} := (A, r, X)$ with

$$\begin{aligned} A &= \tilde{A} / \sim; \\ X &= \{(x_j)_{j \in I} \mid \forall j \in I, x_j \in X_j \text{ \& } \varphi_{i,j}^-(x_j) = x_i \text{ for any } i \leq j\}; \\ r([\{a_i\}_{i \geq i_0}], (x_j)_{j \in I}) &= r_{i_0}(a_{i_0}, x_{i_0}); \\ \pi_i : \mathcal{C}_i &\rightarrow \mathcal{C} \text{ is: } \pi_i^+(a) = [\{a_j\}_{j \geq i}] \text{ with } a_i = a; \pi_i^-(\tilde{x}) = x_i \text{ for any } \tilde{x} \in X. \end{aligned}$$

One can check that r and π_i 's are well defined, and $\{\pi_i : \mathcal{C}_i \rightarrow \mathcal{C}\}_{i \in I}$ forms a cone in $i\mathbf{C}$. Then along with Construction 3.13 and similar arguments in [4], one can prove all main results in [4] and the results we will present subsequently, with a single exception in [4]: the existence of colimit in $i\mathbf{C}_{bif}$. The proof for the ω -sequence case in $i\mathbf{C}_{bif}$ employed König's Lemma which holds for countable finitely branching trees but not for all larger cardinalities.

In our present definition of E-bifinite spaces, we could also use arbitrary co-directed systems instead of ω -sequences. However, we prefer to work with the present definition as it stresses the computability aspect: the E-bifinite spaces can be constructed from strongly finite ones by a countable process.

4 The tensor product in E-bifinite Chu spaces

In this section we show that \mathbf{E}_{bif} and $i\mathbf{E}_{bif}$ are monoidal categories. First we recall the classical definition of monoidal categories ([5]).

Definition 4.1 A monoidal category \mathbf{A} is a category equipped with a bifunctor $\otimes : \mathbf{A} \times \mathbf{A} \rightarrow \mathbf{A}$ called the tensor product, and an object I called the unit object, such that

- (1) \otimes is associative with respect to natural isomorphisms;
- (2) \otimes has I as left and right identity;
- (3) for any objects A, B, C and D in \mathbf{A} , the diagram

$$\begin{array}{ccc} ((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha_{A,B,C} \otimes D} & (A \otimes (B \otimes C)) \otimes D \xrightarrow{\alpha_{A,B \otimes C,D}} A \otimes ((B \otimes C) \otimes D) \\ \alpha_{A \otimes B,C,D} \downarrow & & \downarrow A \otimes \alpha_{B,C,D} \\ (A \otimes B) \otimes (C \otimes D) & \xrightarrow{\alpha_{A,B,C \otimes D}} & A \otimes (B \otimes (C \otimes D)) \end{array}$$

commutes, where α is the corresponding natural isomorphism;

- (4) for any objects A and B in \mathbf{A} , the diagram

$$\begin{array}{ccc} (A \otimes I) \otimes B & \xrightarrow{\alpha_{A,I,B}} & A \otimes (I \otimes B) \\ \rho_{A \otimes B} \searrow & & \swarrow A \otimes \lambda_B \\ & A \otimes B & \end{array}$$

commutes, where ρ and λ denote the corresponding natural isomorphisms from $A \otimes I$ to A , $I \otimes B$ to B , respectively, and α the natural isomorphism with respect to the associative law.

With the tensor product and unit Chu space \mathcal{I} introduced in Section 2, we know that the category \mathbf{C} and \mathbf{E} are monoidal categories. Next we show \mathbf{E}_{bif} and $i\mathbf{E}_{bif}$ are monoidal categories as well.

Proposition 4.2 *If Σ is finite, then \mathcal{I} is strongly finite and thus E -bifinite.*

Proof. It follows directly from Theorem 3.8. \square

Proposition 4.3 *The tensor product of two strongly finite Chu spaces is strongly finite.*

Proof. Let $\mathcal{A} = (A, r, X)$, $\mathcal{B} = (B, s, Y)$ be two strongly finite Chu spaces. Then $\mathcal{A} \otimes \mathcal{B}$ is extensional since so are both \mathcal{A} and \mathcal{B} . In addition, $\text{obj}(\mathcal{A} \otimes \mathcal{B})$ and $\text{attr}(\mathcal{A} \otimes \mathcal{B})$ are finite. So $\mathcal{A} \otimes \mathcal{B}$ is strongly finite. \square

Proposition 4.4 *Let $(\mathcal{A}_i, \varphi_i)_{i \geq 1}$, $(\mathcal{B}_i, \psi_i)_{i \geq 1}$ be two ω -sequences of strongly finite Chu spaces, where $\mathcal{A}_i = (A_i, r_i, X_i)$, $\mathcal{B}_i = (B_i, s_i, Y_i)$ for each $i \geq 1$. Let $\mathcal{A} = (A, r, X) = \text{colim}_{i \geq 1} \mathcal{A}_i$ and $\mathcal{B} = (B, s, Y) = \text{colim}_{i \geq 1} \mathcal{B}_i$ be the colimit of $(\mathcal{A}_i, \varphi_i)_{i \geq 1}$ and $(\mathcal{B}_i, \psi_i)_{i \geq 1}$ respectively. Then $\text{colim}_{i \geq 1} (\mathcal{A}_i \otimes \mathcal{B}_i)$ exists, and is isomorphic to $\mathcal{A} \otimes \mathcal{B}$.*

Proof. Without loss of generality, assume all φ_i^+ 's and ψ_i^+ 's to be inclusion mappings, and let $\mathcal{C}_i = \mathcal{A}_i \otimes \mathcal{B}_i$ and $\eta_i = \varphi_i \otimes \psi_i : \mathcal{C}_i \rightarrow \mathcal{C}_{i+1}$ for $i \geq 1$. Note that \mathcal{C}_i 's are strongly finite and η_i 's are monic, so $\{(\mathcal{C}_i, \eta_i)\}$ is an ω -sequence of strongly finite Chu spaces with colimit $(\mathcal{C}, \{\alpha_i : \mathcal{C}_i \rightarrow \mathcal{C}\}_{i \geq 1})$ as given in Construction 3.3. We also assume \mathcal{A} and \mathcal{B} are given as in Construction 3.3 as well.

Now define $\delta^+ : \text{obj}(\mathcal{A} \otimes \mathcal{B}) \rightarrow \text{obj}(\mathcal{C})$ by $\delta^+((a, b)) = (a, b)$ for any $(a, b) \in \text{obj}(\mathcal{A} \otimes \mathcal{B})$. Since $\{A_i\}_{i \geq 1}$ and $\{B_i\}_{i \geq 1}$ are non-decreasing sequence of set, $\text{obj}(\mathcal{A}) = (\bigcup_{i \geq 1} A_i) \times (\bigcup_{i \geq 1} B_i) = \bigcup_{i \geq 1} (A_i \times B_i) = \text{obj}(\mathcal{C})$. Thus δ^+ is a bijection.

On the other hand, define a mapping $\delta^- : \text{attr}(\mathcal{C}) \rightarrow \text{attr}(\mathcal{A} \otimes \mathcal{B})$ by

$$\delta^-((f_i, g_i)_{i \geq 1})(a, b) = \{(y_i, x_i)\}_{i \geq 1}$$

for any $(f_i, g_i)_{i \geq 1} \in \text{attr}(\mathcal{C})$ and $(a, b) \in \text{obj}(\mathcal{A} \otimes \mathcal{B})$, where, under the assumption of both $a \in A_{i_0}$ and $b \in B_{i_0}$ for some $i_0 \geq 1$, $y_i = f_i(a)$ and $x_i = g_i(b)$ when $i \geq i_0$, and inductively $y_i = \psi_i^-(y_{i+1})$, $x_i = \varphi_i^-(x_{i+1})$ for $1 \leq i \leq i_0 - 1$. Note that for any $\{(f_i, g_i)\}_{i \geq 1} \in \text{attr}(\mathcal{C})$, $(f_i, g_i) = (\varphi_i \otimes \psi_i)^-((f_{i+1}, g_{i+1}))$, that is, $f_i = \psi_i^- \cdot f_{i+1} \cdot \varphi_i^+$ and $g_i = \varphi_i^- \cdot g_{i+1} \cdot \psi_i^+$. Then it can be readily checked that δ^- is well defined. We show δ is a morphism from $\mathcal{A} \otimes \mathcal{B}$ to \mathcal{C} .

In fact, let $s_{\mathcal{A} \otimes \mathcal{B}}$ and $s_{\mathcal{C}}$ denote the satisfaction relation on $\mathcal{A} \otimes \mathcal{B}$ and \mathcal{C} , respectively. Then for any $(a, b) \in \text{obj}(\mathcal{A} \otimes \mathcal{B})$ and $(f_i, g_i)_{i \geq 1} \in \text{attr}(\mathcal{C})$, it can be checked that $s_{\mathcal{C}}(\delta^+(a, b), (f_i, g_i)_{i \geq 1}) = r_{i_0}(f_{i_0}(a), b) = s_{\mathcal{A} \otimes \mathcal{B}}((a, b), \delta^-((f_i, g_i)_{i \geq 1}))$.

Finally we show that δ is injective and surjective.

For any two distinct points $(f_i, g_i)_{i \geq 1} \in \text{attr}(\mathcal{C})$ and $(f'_i, g'_i)_{i \geq 1} \in \text{attr}(\mathcal{C})$, there is a $j \geq 1$ with $(f_j, g_j) \neq (f'_j, g'_j)$. Then $f_j \neq f'_j$ or $g_j \neq g'_j$, which entails the existence of a point $a \in \text{obj}(\mathcal{A})$ or a point $b \in \text{obj}(\mathcal{B})$ such that $f(a) \neq f'(a)$ or $f(b) \neq g'(b)$. So $\delta^-((f_i, g_i)_{i \geq 1})(a, b) \neq \delta^-((f'_i, g'_i)_{i \geq 1})(a, b)$. Thus we know that δ^- is injective.

Now for any $(f, g) \in \text{attr}(\mathcal{A} \otimes \mathcal{B})$, let $f_i = \pi_i'^- \cdot f \cdot \pi_i^+$ and $g_i = \xi_i^- \cdot g \cdot \pi_i'^+$, where for $i \geq 1$, $\pi_i : \mathcal{A}_i \rightarrow \mathcal{A}$ and $\pi_i' : \mathcal{B}_i \rightarrow \mathcal{B}$ are as defined in the Construction. Then it can be checked that $(f_i, g_i) \in \text{attr}(\mathcal{A}_i \otimes \mathcal{B}_i)$ for all $i \geq 1$, $(f_i, g_i)_{i \geq 1} \in \text{attr}(\mathcal{C})$, and $\delta^-((f_i, g_i)_{i \geq 1}) = (f, g)$. Thus δ^- is surjective. \square

From the two preceding propositions, we have

Corollary 4.5 *The tensor product of two E-bifinite Chu spaces is bifinite.*

Theorem 4.6 *The category \mathbf{E}_{bif} and $i\mathbf{E}_{bif}$ of E-bifinite Chu spaces over finite Σ are monoidal categories.*

Proof. The finiteness of Σ entails that \mathcal{I} is the unit object in \mathbf{E}_{bif} and $i\mathbf{E}_{bif}$. The left is to directly check items (1), (3) and (4) in Definition 3.7. \square

If Σ is infinite, there will be no unit object in the category of E-bifinite Chu spaces, with respect the tensor product defined above. Thus \mathbf{E}_{bif} and $i\mathbf{E}_{bif}$, when over finite Σ , are not monoidal categories any more.

Proposition 4.7 *There is no unit object in \mathbf{E}_{bif} and $i\mathbf{E}_{bif}$ if Σ is infinite.*

Proof. Assume that $\mathcal{J} = (D, t, Z)$ is a unit object in \mathbf{E}_{bif} and $i\mathbf{E}_{bif}$. We show that Σ must be finite. Since for any E-bifinite Chu space \mathcal{A} , $\mathcal{A} \otimes \mathcal{J} \cong \mathcal{A}$ implies $|\text{obj}(\mathcal{A} \otimes \mathcal{J})| = |\text{obj}(\mathcal{A})|$ by Corollary 2.3, $|D|$ must be 1. Let $D = \{d_0\}$.

Further, the attribute set Z should be finite. In fact, since \mathcal{J} is bifinite, there exists an ω -sequence of $\{\varphi_i : \mathcal{C}_i \rightarrow \mathcal{C}_{i+1}\}_{i \geq 1}$ with $\psi_i : \mathcal{C}_i \rightarrow \mathcal{J}$ as its colimit, where $\mathcal{C}_i = (C_i, r_i, X_i)$ is strongly finite for $i \geq 1$. Since $|\text{obj}(\mathcal{J})|=1$, from Construction 3.3 we know that $|C_i| = 1$ for any $i \geq 1$. Let $C_i = \{c_i\}$ for $i \geq 1$. Note that for any $z \in Z$, we have $r_1(c_1, \psi_1^-(z)) = t(\psi_1^+(c_1), z) = t(d_0, z)$. Since \mathcal{J} is extensional, different choice of z will lead to different $t(d_0, z)$ and thus different $\psi_1^-(z)$, that is, ψ_1^- is an injection. Then the finiteness of X_1 entails that of Z .

Finally, $t : D \times Z \rightarrow \Sigma$ is a bijection and thus Σ is finite. In fact, since \mathcal{J} is extensional, t is an injection and thus $|Z| \leq |\Sigma|$. Assume that t is not a surjection and let $z_0 \in \Sigma - t(D \times Z)$. Then let the Chu space $\mathcal{A} = (\{a\}, r, \{b\})$ where $r : \{a\} \times \{b\} \rightarrow \Sigma$ satisfies $r(a, b) = z_0$. It can be readily checked that $\mathcal{A} \otimes \mathcal{J} \not\cong \mathcal{A}$ since $\text{attr}(\mathcal{A}) = 1$ while $\text{attr}(\mathcal{A} \otimes \mathcal{J}) = 0$. A contradiction to the assumption that \mathcal{J} is a unit object. \square

5 Non-monoidal-closedness of \mathbf{E}_{bif} and $i\mathbf{E}_{bif}$

We will discuss about the non-monoidal-closedness of E-bifinite Chu spaces in this section. First, we recall the notion of monoidal closedness ([5]).

Definition 5.1 A monoidal category \mathbf{A} with a tensor product \otimes is closed if there is a bifunctor $\multimap: \mathbf{A}^{op} \times \mathbf{A} \rightarrow \mathbf{A}$ such that for any objects B and C of \mathbf{A} , the functor $A \multimap -$ is left adjoint to $A \otimes -$, namely, $\text{Hom}(A \otimes B, C) \cong \text{Hom}(B, A \multimap C)$ under natural isomorphisms in the category **SET**.

Without too much effort, one can show that the linear operator introduced in section 2 will lead the category \mathbf{C} to monoidal closedness. However, such an operator is not appropriate to make \mathbf{E}_{bif} or $i\mathbf{E}_{bif}$ monoidal closed.

Example 5.2 The linear operator \multimap is not a functor on neither \mathbf{E}_{bif} nor $i\mathbf{E}_{bif}$. In fact, consider the Chu space $\mathcal{A} = (A, r, X)$ on $\Sigma = \{0, 1\}$, where $A = X = \{1, 2\}$ and $r(a, x) = 1$ if $a \leq x$ and $r(a, x) = 0$ otherwise. Then $\text{Hom}(\mathcal{A}, \mathcal{A}) = \{\varphi = (c_1, c_2), \psi = (id, id)\}$, where c_1 and c_2 are the constant mapping from $\{1, 2\}$ to 1 and 2, respectively, and id is the identity mapping on $\{1, 2\}$. And the satisfaction relation $t: \text{Hom}(\mathcal{A}, \mathcal{A}) \times (A \times X) \rightarrow \Sigma$ is:

t	$(1, 1)$	$(1, 2)$	$(2, 1)$	$(2, 2)$
φ	1	1	1	1
ψ	1	1	0	1

From this table, we can observe that $\mathcal{A} \multimap \mathcal{A}$ is not extensional and so is not strongly finite. Thus the linear operator is not closed in the categories \mathbf{E}_{bif} nor $i\mathbf{E}_{bif}$.

Further, consider the operator $\&$ defined in [3] as: $\mathcal{A} \& \mathcal{B} = \mathcal{A}^\perp \multimap \mathcal{B}$. With the same \mathcal{A} in the preceding example, we can observe that $\&$ is still not a functor on neither \mathbf{E}_{bif} nor $i\mathbf{E}_{bif}$. In fact, we believe that when assigned with the tensor product of Definition 2.4, the category of E-bifinite Chu spaces will never be closed with respect to any linear operator on it. So we give the following claim.

Claim 5.3 Neither \mathbf{E}_{bif} nor $i\mathbf{E}_{bif}$ can be monoidal closed with respect to the tensor product in Definition 2.4.

6 Conclusion

In [4], Droste and Zhang introduced the notion of bifinite Chu spaces and studied a number of their interesting properties. In this paper, we continue their work by introducing a new type of bifinite Chu spaces called E-bifinite Chu spaces. We have shown that the subcategories \mathbf{E}_{bif} and $i\mathbf{E}_{bif}$ of Chu spaces are monoidal when Σ is finite. However, neither \mathbf{E}_{bif} nor $i\mathbf{E}_{bif}$ is monoidal closed with respect to the standard tensor product and linear operator. We believe, although have not yet proven, that there are no linear operators which would make the respective categories monoidal closed using the standard tensor product.

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